Base–structure interaction of linearly isolated structures with lateral–torsional coupling

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Abstract

The linear earthquake response of seismically isolated structures with lateral–torsional coupling is investigated. Emphasis is placed on developing simplified procedures for estimating the amplification of edge displacements of the superstructure and isolated base. The three-dimensional response of asymmetric buildings is cast under a dynamic base–superstructure interaction formulation. Among the wide range of possibilities to represent this interaction, two simplified models were selected. The first model accounts for the base–superstructure interaction through a correction of the mass matrix of the superstructure, while the second assumes a pseudo-static response of the superstructure subject to three lateral inertial force distributions. Symbolic expressions are derived to compute the edge response by using the pseudo-static method. Such expressions are simple to use and show better accuracy than the one implicit in current isolation codes. Finally, the response of a six-story asymmetric building example demonstrates the application of the proposed procedure, and results are compared with the true peak response at both edges computed from integration of the equations of motion of the isolated structure.

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1. Introduction

Lateral–torsional coupling in seismically isolated structures results from plan asymmetry in the superstructure as well as in the isolation base. Indeed, since the equations of motion of the base and superstructure are coupled, any plan asymmetry present in either subsystem creates lateral–torsional coupling in the combined isolated structure. Such coupling leads to an uneven displacement demand across the building plan, causing some isolators and structural members to experience larger deformations and forces than others. Because experience has shown that seismic isolation may be used in conjunction with asymmetric superstructures, it is important to derive design expressions to estimate the expected torsional amplifications of the edge response for the base and superstructure. This article emphasizes the torsional response of the base.

Although most isolation devices show inelastic force–deformation behavior, it is possible to model their effect on a structure by using linear equivalent models. Beyond their use as a preliminary design tool, linear equivalent models capture the essence of the seismic response of an isolated structure. For example, the variation of the isolated period of a structure with elastomeric bearings for shear deformations in the range $\gamma = 100 \rightarrow 250\%$, relative to a target period corresponding to $\gamma = 150\%$, ranges between $\pm 3\%$ [1]. This small variation in period warrants the use of linear equivalent models for global response estimations.

The early work of Pan and Kelly [9] concluded that seismic isolation led naturally to torsional balance; a rigid superstructure with isolated torsional-to-lateral frequency ratio $\Omega_b = 1$ was considered in this study. Lee [5] reached a similar conclusion considering a rigid superstructure, rectangular in plan, supported on non-linear isolators and subjected to ground motion records; the structural model considered also had $\Omega_b = 1$ and the isolators and masses were placed at the corners of the building plan. Later, Nakamura et al. [8] derived similar conclusions by using experimental results from shaking table experiments on a single-story asymmetric plan structure.
authors evaluated the influence of the superstructure on the isolation system.

Although restricted to very small amplitude motions, measured responses in the Law and Justice Center building (Rancho Cucamonga) during the 1985 Redlands earthquake showed torsional amplifications [10]. Motivated by this observation, Nagarajaiah et al. [7] studied the inelastic response of seismically isolated asymmetric plan multistory buildings subjected to recorded ground motions. The isolation system considered was inelastic and several torsional-to-lateral frequency ratios and eccentricities for multistory buildings were considered. They concluded that in some cases the isolation system may amplify considerably the edge response of the structure. Kulkarni and Jangid [6] evaluated the effect of the superstructure flexibility for buildings isolated with linear and non-linear bearings, and concluded that the response of the isolation system is not greatly influenced by this flexibility. However, the effect in total floor accelerations increases as the flexibility of the fixed-base structure and the restoring force of the isolation system also increase. More recently, in a different line of work Ryan and Chopra [12] developed simplified methods in the linear range to account for torsional effects in the isolation system due to the effects of plan asymmetry in the superstructure.

Although with a similar final goal, the approach followed in this investigation is somewhat different from that used in previous work [5,8,9,12]. It is based on the interpretation of the dynamic torsional interaction between the isolated base and superstructure. The understanding of such interaction confines the results of previous researchers within a unique framework. Moreover, it touches also on the concept of lateral–torsional passive control of the isolated base by the use of linear isolation. Because the formulation of the equations of motion based on relative coordinates is well known [7], the emphasis of this study is on the interpretation of the dynamic interaction that occurs between the base and superstructure and how the superstructure perceives the motion of the base and vice versa. The extension of the formulation presented herein to the case of inelastic and geometrically non-linear isolators is straightforward.

Due to the interest in deriving a simple procedure to estimate the edge displacement amplifications, the emphasis is placed on simple modal and static procedures. Such procedures are applied to monosymmetric and fully asymmetric structures. As usual, the control of the torsional response of the structure is done by controlling the edge displacements of the building plan. The article concludes by applying the simplified procedure to estimate the torsional amplification at the isolated base and superstructure of a six-story building example.

2. Structural model and equations of motion

The asymmetric multistory structural system considered in this investigation is schematically shown in Fig. 1. The model considered has no restriction on the extent or type of plan asymmetry (mass or stiffness). Structures are subjected to two horizontal components of ground motion denoted as $\mathbf{u}' = \mathbf{u} + \mathbf{r}^{(s)} \mathbf{q}'$ and $\mathbf{q}' = \mathbf{q} + \mathbf{r}^{(b)} \mathbf{u}_g$

where $\mathbf{r}^{(b)}$ represents the displacements that would occur in the isolated base as a result of unitary ground displacements; and $\mathbf{r}^{(s)}$ the displacements that would occur in the superstructure as a result of unitary displacements of the isolated base. Eq. (1) may be stated in a more compact form as the following upper triangular kinematic transformation

$$
\mathbf{x} = \begin{bmatrix}
\mathbf{u}'
\mathbf{q}'
\mathbf{u}_g
\end{bmatrix} = \begin{bmatrix}
1 & \mathbf{r}^{(s)} & \mathbf{r}^{(b)}
0 & 1 & \mathbf{r}^{(b)}
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}
\mathbf{q}
\mathbf{u}_g
\end{bmatrix} = \mathbf{Lx}
$$

where $\mathbf{x}$ and $\mathbf{x}$ represent the vectors of relative and total displacements, respectively. Substituting Eq. (2) into the
equations of motion of the structure leads to
\[
\begin{bmatrix}
\mathbf{m}^{(s)} & \mathbf{m}^{(s)}
\end{bmatrix} \begin{bmatrix}
\mathbf{u}^{(s)}
\end{bmatrix} + \begin{bmatrix}
\mathbf{c}^{(s)} & \mathbf{0}
\end{bmatrix} \begin{bmatrix}
\dot{\mathbf{u}}^{(s)}
\end{bmatrix} = -\begin{bmatrix}
\mathbf{r}^{(s)}
\end{bmatrix}^{T} \mathbf{m}^{(s)} \begin{bmatrix}
\mathbf{u}^{(s)}
\end{bmatrix}
\]
\[
\cdots + \begin{bmatrix}
\mathbf{k}^{(s)} & \mathbf{0} & \mathbf{k}^{(s)} & \mathbf{0}
\end{bmatrix} \begin{bmatrix}
\mathbf{u}
\end{bmatrix}
= -\begin{bmatrix}
\mathbf{m}^{(s)} \mathbf{r}^{(s)}
\end{bmatrix}^{T} \mathbf{r}^{(b)} \ddot{\mathbf{u}}^{g}
\tag{3}
\]
where \( \mathbf{m}^{(s)} \), \( \mathbf{k}^{(s)} \), and \( \mathbf{c}^{(s)} \) are the mass, stiffness, and damping matrices of the superstructure fixed to the ground; \( \mathbf{m}^{(b)} \), \( \mathbf{k}^{(b)} \), and \( \mathbf{c}^{(b)} \) are the corresponding mass, stiffness, and damping matrices for the isolated base alone; and \( \mathbf{m}^{(l)} = \mathbf{r}^{(s)}^{T} \mathbf{m}^{(s)} \mathbf{r}^{(s)} + \mathbf{m}^{(b)} \) represents the total seismic mass of the structure. In the simple case of vertical alignment of the CM of each floor and a rigid-body kinematic transformation between the motion of the base and superstructure, \( \mathbf{r}^{(s)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} \).

Finally, the rigid-body kinematic transformation between the horizontal motions of the ground and base is \( \mathbf{r}^{(b)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{T} \).

The relative displacements \( \mathbf{u} \) of the superstructure in Eq. (3) could also be expressed in terms of the modes of the fixed-base structure, i.e., the structure without isolation. Thus, if the fixed-base modes and coordinates are \( \Phi \) and \( \mathbf{s} \), respectively, with \( \mathbf{u} = \Phi^T \mathbf{s} \), this new modal transformation leads to
\[
\begin{bmatrix}
\Phi^{T} \mathbf{m}^{(s)} \Phi & \Phi^{T} \mathbf{m}^{(s)} \mathbf{r}^{(s)}
\end{bmatrix} \begin{bmatrix}
\mathbf{s}
\end{bmatrix} + \begin{bmatrix}
\Phi^{T} \mathbf{c}^{(s)} \Phi & \mathbf{0} \\
\mathbf{0} & \mathbf{c}^{(b)}
\end{bmatrix} \begin{bmatrix}
\dot{\mathbf{s}}
\end{bmatrix}
\]
\[
\cdots + \begin{bmatrix}
\Phi^{T} \mathbf{k}^{(s)} \Phi & \mathbf{0} & \Phi^{T} \mathbf{k}^{(s)} \mathbf{r}^{(s)}
\end{bmatrix} \begin{bmatrix}
\mathbf{s}
\end{bmatrix}
= -\begin{bmatrix}
\Phi^{T} \mathbf{m}^{(s)} \mathbf{r}^{(s)}
\end{bmatrix}^{T} \mathbf{r}^{(b)} \ddot{\mathbf{u}}^{g}
\tag{4}
\]
Since \( \Phi \) are the modes of the pencil \( (\mathbf{k}^{(s)}, \mathbf{m}^{(s)}) \), and assuming that the damping matrix \( \mathbf{c}^{(s)} \) is classical, the products \( \Phi^{T} \mathbf{k}^{(s)} \Phi \) and \( \Phi^{T} \mathbf{k}^{(s)} \mathbf{r}^{(s)} \) lead to diagonal matrices. Hence, the coupling between the motions of the base and superstructure occurs exclusively as a result of the off-diagonal terms in the mass matrix. Consequently, the interaction between the motions of the isolated base and superstructure as shown in Eq. (4) may be interpreted as inertial. Notice that the term \( \Phi^{T} \mathbf{m}^{(s)} \mathbf{r}^{(s)} \) represents the projection of the rigid body inertia force terms \( \mathbf{m}^{(s)} \mathbf{r}^{(s)} \), corresponding to a unit base acceleration, on the modal base \( \Phi \).

One of the advantages in casting the equations of motion in terms of relative coordinates is the simplicity of the construction of the non-classical damping matrix of the complete isolated structure. By using the modal decomposition for the superstructure and base, the damping matrices \( \mathbf{c}^{(s)} \) and \( \mathbf{c}^{(b)} \) are
\[
\mathbf{c}^{(s)} = \mathbf{m}^{(s)}^{T} \Phi \mathbf{c}^{(b)} \Phi^{T} \mathbf{m}^{(s)}
\]
\[
\mathbf{c}^{(b)} = \mathbf{m}^{(b)}^{T} \Psi \mathbf{c}^{(b)} \Psi^{T} \mathbf{m}^{(b)}
\tag{5}
\]
where \( \Phi \) are the modes of the pencil \( (\mathbf{k}^{(b)}, \mathbf{m}^{(b)}) \); \( \mathbf{c}^{(s)} = \text{diag}[2\xi_{i}^{(s)} \omega_{i}^{(s)}/\rho_{i}^{(s)}] \) \((i = 1 : 3N)\) and \( \mathbf{c}^{(b)} = \text{diag}[2\xi_{j}^{(b)} \omega_{j}^{(b)}/\rho_{j}^{(b)}] \) \((j = 1 : 3)\) are the modal damping matrices of the superstructure and base; \( \mathbf{m}^{(s)} = \Phi^{T} \mathbf{m}^{(s)} \Phi \) and \( \mathbf{m}^{(b)} = \Psi^{T} \mathbf{m}^{(b)} \Psi \) are the corresponding modal mass matrices; \( \xi_{i}^{(s)} \), \( \omega_{i}^{(s)} \), \( \xi_{j}^{(b)} \), and \( \omega_{j}^{(b)} \) are the damping ratios and coupled frequencies of the superstructure with fixed base and those for the base with the superstructure acting as a rigid body. In this article, constant modal damping ratios of 5% and 15% will be assumed for the superstructure and base, respectively [1]. Notice that this formulation may be conveniently extended to include inelastic and geometrically non-linear isolation systems [4,14].

3. Base–structure interaction problem

One approach to understand the three-dimensional response of a seismically isolated structure is to rewrite Eq. (3) or (4) as an interaction problem of two subsystems by simply moving to the right hand side of the equations the coupling terms of the mass matrix,
\[
\begin{bmatrix}
\mathbf{m}^{(s)} & \mathbf{0} & \mathbf{0}
\end{bmatrix} \begin{bmatrix}
\ddot{\mathbf{u}}^{(s)}
\end{bmatrix} + \begin{bmatrix}
\mathbf{c}^{(s)} & \mathbf{0} & \mathbf{0}
\end{bmatrix} \begin{bmatrix}
\dot{\mathbf{u}}^{(s)}
\end{bmatrix}
\]
\[
\cdots + \begin{bmatrix}
\mathbf{k}^{(s)} & \mathbf{0} & \mathbf{k}^{(s)} & \mathbf{0}
\end{bmatrix} \begin{bmatrix}
\mathbf{q}
\end{bmatrix}
= -\begin{bmatrix}
\mathbf{m}^{(s)} \mathbf{r}^{(s)} \ddot{\mathbf{u}}^{g}
\end{bmatrix}^{T} \mathbf{r}^{(b)} \ddot{\mathbf{u}}^{g}
\tag{6}
\]
As was stated before, the dynamic interaction between the two subsystems may be represented through the inertia terms. For a typically isolated structure with a good separation between the fixed-base and isolated periods, a good approximation for the dynamic response of the base would be obtained by assuming a rigid superstructure. If such an assumption is mathematically enforced it implies that \( \ddot{\mathbf{u}} = \mathbf{0} \) in the right hand side of the second block of Eq. (6), leading to the approximate uncoupled motion of the base
\[
\mathbf{m}^{(l)} \ddot{\mathbf{q}}^{\infty} + \mathbf{c}^{(b)} \dot{\mathbf{q}}^{\infty} + \mathbf{k}^{(b)} \mathbf{q}^{\infty} = -\mathbf{m}^{(l)} \mathbf{r}^{(l)} \ddot{\mathbf{u}}^{g}
\tag{7}
\]
where \( \ddot{\mathbf{q}}^{\infty}, \dot{\mathbf{q}}^{\infty}, \) and \( \mathbf{q}^{\infty} \) are the base acceleration, velocity, and displacement, assuming a rigid superstructure. Although an obvious observation, Eq. (7) explicitly shows that the lateral–torsional characteristics of the superstructure are not that essential since the interaction of the superstructure into the base decreases as the superstructure becomes stiffer. Contrarily, the interaction effects of the base into the superstructure are always significant and will enable us to control in most cases the lateral–torsional coupling effects of the superstructure by changing the properties of the isolation system [13].

An iterative predictor–corrector strategy to exactly integrate Eq. (6) is as follows. A predictor for the relative acceleration \( \ddot{\mathbf{q}} \) of the base is obtained by first integrating Eq. (7). This predicted base acceleration is introduced into the first block of Eq. (6) to compute a first estimation of \( \ddot{\mathbf{u}} \). The estimated superstructure acceleration vector \( \ddot{\mathbf{u}} \) is introduced into the second block of Eq. (6) to compute a corrected value for the base acceleration \( \ddot{\mathbf{q}} \). The iteration then continues alternating the solution between these two blocks of Eq. (6). This predictor–corrector strategy works extremely well and leads to
the exact response in very few iterations. This model is denoted as the exact model in Fig. 2 (EXM), $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Nevertheless, the base and superstructure dynamic interaction present in Eq. (6) may still be interpreted somewhat differently. Before attempting to evaluate different approximation levels to account for the flexibility of the superstructure, it is useful to solve for the base acceleration $\mathbf{q}$ from the second block of Eq. (6) and substitute this value into the first block of Eq. (6). This substitution leads to an exact dynamic equation for the response of the superstructure in terms of the displacement $\mathbf{q}$ and velocity $\dot{\mathbf{q}}$ of the isolated base

$$\mathbf{m}^{(s)} \ddot{\mathbf{u}} + \mathbf{c}^{(s)} \dot{\mathbf{u}} + \mathbf{k}^{(s)} \mathbf{u} = \mathbf{m}^{(s)} \mathbf{r}^{(s)} \mathbf{H}^{\infty} \mathbf{q}$$

where $\mathbf{m}^{(s)} = \mathbf{m}^{(s)} - \mathbf{m}^{(s)} \mathbf{r}^{(s)} \mathbf{m}^{(s)} \mathbf{r}^{(s)} \mathbf{r}^{(s)}$ is interpreted as a corrected mass matrix for the superstructure; and $\dot{\mathbf{q}}^{(s)} = -\mathbf{m}^{(s)} \mathbf{r}^{(s)} \mathbf{q} + \mathbf{k}^{(s)} \mathbf{q}$ is the input for the superstructure and may be interpreted as the total base acceleration associated with a rigid behavior of the superstructure. Indeed, if a rigid superstructure is considered in computing $\dot{\mathbf{q}}$ and $\mathbf{q}$, a model approximation is introduced and the mass-corrected model (MCM) is denoted as $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = 1$ in Fig. 2. Please notice that the dynamic interaction between the base and structure is perceived by the superstructure as a reduction on its dynamic mass (inertial interaction), and that the input for the superstructure $\dot{\mathbf{q}}^{(s)}$ will be dominated by the three long-period motions of the isolated base. Indeed, these motions will be controlled by frequencies $\omega_j^b$, and the base modes corresponding to a rigid superstructure, $\Psi$.

Because the superstructure becomes even stiffer due to the interaction effect represented by the mass correction, and the excitation to the superstructure is a low-frequency motion, it is intuitively appealing to propose a simplified model that neglects the dynamic response of the superstructure and assumes that the problem is that of a statically imposed base motion. The quasi-static model (QSM) is denoted as $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in Fig. 2 and may be stated by the static algebraic relationship

$$\mathbf{k}^{(s)} \mathbf{u} = -\mathbf{m}^{(s)} \mathbf{r}^{(s)} \dot{\mathbf{q}}^{(s)}_\infty$$

Notice that the term $\mathbf{m}^{(s)} \mathbf{r}^{(s)} \dot{\mathbf{q}}^{(s)}_\infty$ is interpreted as the floor inertia forces on the superstructure for a rigid-body type acceleration field of the base.

Shown in the block diagram of Fig. 2 are the three structural models just described, EXM, MCM, and QSM, represented by the different values of the $\alpha_1$, $\alpha_2$, and $\alpha_3$ parameters. Several other approximations for integrating the equations of motion may be introduced at this point; however, the three just presented are significant in terms of their physical interpretation. Next, the three models are compared in terms of their numerical accuracy and range of applicability.

Consider the frequency response functions (FRF) of a six-story isolated building with plan aspect ratio $a/b = 2$ subjected to a single component ground motion $u_{gx}(t)$ shown in Fig. 3. The period, damping ratio, stiffness eccentricity, and uncoupled torsional-to-lateral frequency ratio are $T_s = 0.4$ s, $\xi_s = 0.05$, $e_{sx} = 0.25r$—where $\rho$ is the mass radius of gyration, and $\Omega_b = 1.2$ for the fixed-base structure, and $T_h = 2.5$ s, $\xi_h = 0.15$, $e_{bh} = 0$, and $\Omega_h = 1$ for the isolated base with rigid superstructure. Fig. 3 compares the moduli of the FRF for the roof drift $|H_{\text{drift}}(j\omega)|$ and acceleration $|H_{\text{acc}}(j\omega)|$ at the flexible and stiff edges obtained for the three models considered. It is apparent that the EXM and MCM are essentially identical for all frequency ranges considered in the analysis. This implies that the correction in mass included in the MCM captures extremely well the dynamic interaction between the base and superstructure. On the other hand, for response quantities with significant frequency content, say less than 2–3 Hz, the QSM provides a reasonable approximation, especially in terms of the interstory drifts of the superstructure.

As should be expected, given the larger participation of high frequency harmonics in the roof accelerations shown in Fig. 3, the range of application of the QSM decreases in this case, but it remains quite ample.

Roof drift and acceleration histories for the flexible and stiff edges of the previously described six-story building subjected to the Newhall record are presented in Fig. 4. The solid, dash-dot, and dash traces correspond to the EXM, MCM, and QSM earthquake responses, respectively. The two approximate models, MCM and QSM, are capable of accurately predicting the roof drift histories that differ by a factor of 2 between edges as a result of lateral–torsional coupling present in the superstructure. Although not presented in the figure, the accuracy of the predicted response for the isolated base response is always better than that for the superstructure. In this case, the QSM drift error obtained at the instant of maximum drift value is less than 4.5% and occurs at the flexible edge. On the other hand, QSM prediction errors tend to increase when considering edge accelerations and larger discrepancies are apparent from the traces shown in part (b) of Fig. 4 as a result of higher mode contributions that are filtered by the QSM. Just as a reference, the QSM error at the instant of peak accelerations at both edges is less than 7%. Furthermore, the MCM provides excellent accuracy for the edge drifts and accelerations of the superstructure.

A wider range of six-story building responses is analyzed and presented in terms of model error spectra in Fig. 5. Results
are presented for bidirectional input and correspond to mean values obtained for eight recorded ground motions (Table 1). The abscissae contain the fixed-to-isolated period ratio of a structure, $T_s/T_b$, and the ordinates are the errors measured at the peak response. Geometric and dynamic properties of the six-story isolated structure are $a/b = 2$, $\xi_s = 0.05$, $\epsilon_{sx} = 0.20\rho$, $T_b = 2.5$ s, $e_{bx} = 0$, $\Omega_b = 1$, and $\zeta_b = 0.15$ for the superstructure, ranging from torsionally flexible to torsionally stiff buildings. The first column plots show the mean error in base drifts at the flexible and stiff edge; the second and third column plots contain the roof-to-base drifts at the flexible and stiff edges, respectively. In general, the errors increase with increasing period ratio $T_s/T_b$. Also, while the MCM tends to overestimate the response of the base and superstructure, i.e., it is on the conservative side, the QSM tends to underestimate such a response. Moreover, model prediction errors tend to be larger for torsionally flexible than for torsionally stiff superstructures. The model prediction error for the isolated base drift is smaller at the stiff edge, and for both edges is less than 10% if the structure has a period ratio...
corresponds to three low frequency motions that are approximate since it assumes a modal coordinate response of the system described by Eqs. (7) and (8). After some basic algebraic manipulation, it can be shown that

\[ \ddot{q}_\infty = \Psi \ddot{\eta} + R \dddot{u} = -\Psi (\Lambda^2 \eta + \Theta \dddot{\eta}) \]

where \( \Lambda^2 = \text{diag}(\omega_k^2) \) and \( \Theta = \text{diag}(2\xi_k\omega_k) \) with \( k = 1, 2, 3 \). Recall that Eq. (10) is approximate since it assumes a rigid superstructure. Notice also that the total base acceleration \( \ddot{q}_\infty \) is dominated by the low frequency modes of the isolated base. Therefore, the term \(-m^{(s)}(s)k_\infty\) may be interpreted as the inertial forces applied to the degrees of freedom of the superstructure corresponding to a total acceleration \( \ddot{q}_\infty \) of the base.

By introducing into Eq. (8) the modal coordinate transformation \( u = \Phi s \), where \( \Phi \) are the modes of the pencil \( (k^{(s)}, m^{(s)}) \), the modal response of the superstructure may be written as

\[ m^{(s)} \ddot{s} + c^{(s)} \dot{s} + k^{(s)} s = -\Phi^T m^{(s)}(s) \dddot{q}_\infty \]

where \( \Phi^{(s)} = \Phi^T m^{(s)} \Phi \) and \( c^{(s)} \) are diagonal matrices; and \( k^{(s)}(s) \) is not strictly diagonal, but predominantly so. Since \( \ddot{q}_\infty \) is known from Eq. (10), the integration of Eq. (11) is a conventional modal analysis (if \( c^{(s)} \) is treated as classical) that requires the integration in time of 3N single degree of freedom equations. Perhaps the only distinctive aspect is that the input of Eq. (11) corresponds to three low frequency motions that are eventually coupled by the inherent asymmetry of the isolation system.

Because of the low frequency content of the input and the good accuracy shown by the QSM in the previous section, it is possible to propose an estimation of the expected maximum response of the superstructure by using Eq. (9) instead of (11), i.e.,

\[ u(t) = -k^{(s)} \dddot{q}_\infty = f^{(s)} R \]

where \( f^{(s)} \) represents the flexibility matrix of the superstructure, and \( R_k = -m^{(s)}(s) \dddot{q}_\infty \) the inertia force vector due to the total base acceleration \( \dddot{q}_\infty \). Thus, the response of the superstructure may be expressed as

\[ u(t) = f^{(s)}[m^{(s)}(s) \Psi(\Lambda^2 \eta + \Theta \dddot{\eta})] = \sum_{k=1}^{3} f^{(s)} R_k \]

\[ = \sum_{k=1}^{3} u_k(t) \]

where \( R_k = (\omega_k^2 \eta_k(t) + 2\xi_k\omega_k \eta_k(t))m^{(s)} \dddot{q}_\infty \) is the vector of floor inertia forces induced by the kth modal base acceleration on a rigid superstructure; and \( u_k(t) = f^{(s)} R_k \) is the time varying displacement of the superstructure due to the kth vibration mode of the isolated base.

It is now possible to apply spectral superposition on Eq. (13) in order to estimate the expected value of, say, the displacement response of the superstructure—or any other response required.

Let us define

\[ r_k(t) = E \left[ \max_{t \leq \tau} |u_k(t)| \right] \]
where $E[\cdot]$ represents the expected value of $\cdot$ on the time window of length $\tau$; and $r_{k\alpha}$ is the expected value of the maximum displacement vector of the superstructure. By assuming zero correlation between $\eta_k(t)$ and $\bar{\eta}_k(t)$ and using the harmonic approximation $E[\bar{\eta}_k(t)^2] = \omega_k^2 E[\eta_k(t)^2]$, Eq. (14) could be simplified as

$$r_{k\alpha} = \frac{d_k}{\sqrt{1 + 4\zeta_k^2 \omega_k^2}} \left[ \max_{t \leq \tau} |\eta_k(t)| \right]$$

$$= d_k \sqrt{1 + 4\zeta_k^2 \omega_k^2} A_k \sqrt{1 + 4\zeta_k^2 \omega_k^2} A_k^\dagger$$

where $d_k = f^{(k)} m^{(k)} r^{(k)} \bar{\eta}_k$ is the spatial component of the modal displacement $\bar{u}_k(t)$; $L_k^{(b)} = -\bar{q}_k^{(b)} m^{(b)} r^{(b)}$ is the $k$th modal participation factor for the isolated structure; $\Gamma_k = L_k^{(b)} / m_k^{(b)}$; $D_k^{(b)}$ and $A_k^{(b)} = \omega_k^{(b)^2} D_k^{(b)}$ are the $k$th spectral displacement and the corresponding spectral pseudo-acceleration, respectively; and $\bar{A}_k^{(b)} = \sqrt{1 + 4\zeta_k^2 \omega_k^2} A_k^{(b)}$. Consequently, any maximum building response $R_o$ may be estimated by using, say, the conventional CQC [2] rule (using the index summation convention)

$$R_o = \max_{t \leq \tau} |R(t)| = (R_p R_q \rho_{pq})^{1/2}, \quad p, q = 1, 2, 3 \quad (16)$$

where $\rho_{pq}$ is the cross-correlation coefficient of the modal responses for the isolated base [2].

Shown in Fig. 6 is a comparison of the estimation errors as a function of the $T_s / T_b$ period ratio obtained for the base and roof-to-base deformation in three-story isolated buildings by using the QSM (Eq. (16)) and the EXM. Since the excitation is represented by design spectra, UBC [16] and NCh2745 [11], the response for the EXM was computed using the extended non-classical CQC rule described elsewhere [15].

The properties of the isolation system considered are $a/b = 2$, $\zeta_i = 0.05$, $e_{sx} = 0.20\rho$, $\theta_b = 25\text{ s}$, $e_{bx} = 0$, $\zeta_b = 0.15$, and $\Omega_b = 1$. Besides, three values for the torsional-to-lateral frequency ratios of the superstructure were considered in the analysis, $\Omega_e = 0.8, 1, \text{ and } 1.2$. For most $T_s / T_b$ the QSM, in conjunction with Eq. (16), lead to superstructure and base responses with errors that range within ±10%. Model errors tend to be larger for stiff edge responses and increase with increasing $T_s / T_b$; they also tend to be larger for smaller $\Omega_e$. Indeed, the approximate procedure may be applied to all $T_s / T_b < 0.5$ if $\Omega_e = 1.2$ (larger than 1), and to $T_s / T_b < 0.2$ if $\Omega_e = 0.8$ (smaller than 1); in the intermediate range of $\Omega_e$, the limit ratio $T_s / T_b$ of applicability may be considered to vary linearly between 0.2 and 0.5. While the error in response to the UBC and NCh2745 codes shows some notorious similarities, the NCh2745 estimation errors for the base deformation at both edges tend to be larger than the UBC values. Most differences among codes are noticeable for $T_s / T_b > 0.2$.

5. 2D and 3D building response

Based on the above results, it is apparent that the response of the isolated base assuming a rigid superstructure is essential in understanding the response of the superstructure. Consequently, the response of the isolated base is summarized in this section since it has been well established earlier [3]. If Eq. (7) is expanded to the case of an asymmetric-plan structure subjected to two horizontal ground motion components, and the coordinates $\mathbf{q}$ are normalized, the following set of well known parametric equations is obtained

$$\ddot{\mathbf{q}}_\infty + \mathbf{c}^{(b)} \dot{\mathbf{q}}_\infty + \omega_{\infty}^2 \mathbf{q}_\infty = \frac{\alpha}{\Omega_1^2} \ddot{\mathbf{u}}_g + \frac{\alpha \bar{e}_{by}}{\Omega_1^2} \dot{\mathbf{u}}_b + \frac{\alpha^2 e_{by}^2 + e_{bx}^2}{\Omega_1^2}$$

$$= \mathbf{r}^{(b)} \mathbf{u}_g$$

where $\mathbf{q}_\infty = [q_x \quad q_y \quad \rho \rho_q]^T$ is the vector of normalized degrees of freedom; $\rho$ is the radius of gyration of the total mass of the building; $\omega_{\infty}^2 = k^{(b)} / m^t$ is the uncoupled lateral frequency in the $y$-direction; $\alpha = \omega_{\infty}^2 / \omega_{nx}^2$ is the ratio of the squared uncoupled lateral frequencies; $\bar{e}_{bx} = e_{bx} / \rho$ and $\bar{e}_{by} = e_{by} / \rho$ are the normalized static eccentricities in the $x$- and $y$-directions; $\Omega_1 = \omega_{nx} / \omega_{ny}$ is the torsional-to-lateral uncoupled frequency ratio, in which the torsional stiffness is calculated with respect to the center of stiffness (CS) of the base; and $\mathbf{c}^{(b)}$ corresponds to the damping matrix of the base $\mathbf{c}^{(b)}$ divided by the mass $m^t$ and in which the third column and row has been divided by $\rho$. 

---

**Table 1**

<table>
<thead>
<tr>
<th>Soil type</th>
<th>Record</th>
<th>Direction</th>
<th>PGA (g)</th>
<th>PGV (cm/s)</th>
<th>PGD (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>El Centro (1940)</td>
<td>N00E</td>
<td>-0.348</td>
<td>-33.5</td>
<td>-12.4</td>
</tr>
<tr>
<td>II</td>
<td>Melipilla (1985)</td>
<td>NS</td>
<td>-0.687</td>
<td>34.3</td>
<td>13.3</td>
</tr>
<tr>
<td>II</td>
<td>Llo Lleo (1985)</td>
<td>N10E</td>
<td>-0.713</td>
<td>-40.3</td>
<td>-10.8</td>
</tr>
<tr>
<td>II</td>
<td>Corralitos (1989)</td>
<td>N00E</td>
<td>0.630</td>
<td>-55.2</td>
<td>12.0</td>
</tr>
<tr>
<td>II</td>
<td>Sylmar (1994)</td>
<td>N00E</td>
<td>0.843</td>
<td>-128</td>
<td>-30.7</td>
</tr>
<tr>
<td>II</td>
<td>Kobe (1995)</td>
<td>N00E</td>
<td>0.822</td>
<td>81.3</td>
<td>-17.7</td>
</tr>
<tr>
<td>III</td>
<td>Viña del Mar (1985)</td>
<td>S20W</td>
<td>0.363</td>
<td>30.7</td>
<td>-5.53</td>
</tr>
<tr>
<td>III</td>
<td>Arleta (1994)</td>
<td>N00E</td>
<td>0.344</td>
<td>40.4</td>
<td>8.88</td>
</tr>
</tbody>
</table>

Type I: $V_s > 900$ m/s; Type II: 400 m/s $< V_s < 900$ m/s; Type III: $V_s < 400$ m/s.
A couple of simplifications may be introduced into Eq. (17) in order to compute the modes and fundamental vibration frequencies of the 3D system. First notice that for an isolated structure with circular elastomeric bearings $\alpha = 1$, and second, let us define the parameter $\delta = 1 + \Omega_b^2 + \bar{e}_b^2$, where $\bar{e}_b = \sqrt{\bar{e}_{by}^2 + \bar{e}_{bx}^2}$ is the normalized total eccentricity. It can be shown that for the 3D system just described, the normalized coupled vibration frequencies $\tilde{\omega}_j$ are given by

$$\tilde{\omega}_j^2 = (\omega_j/\omega_{oy})^2 = \frac{1}{2} \left( \delta \pm \sqrt{\delta^2 - 4\Omega_b^2} \right), (j = 1)$$

$$\frac{1}{2} \left( \delta \pm \sqrt{\delta^2 - 4\Omega_b^2} \right), (j = 2, 3)$$

where the minus sign in the normalized frequency $\tilde{\omega}_3$ corresponds to the predominantly rotational mode when $\Omega_b < 1$ and to the predominantly translational mode when $\Omega_b > 1$. 

Fig. 6. Errors obtained by using the QSM in conjunction with two $y$-direction design spectra: (a) UBC, and (b) NCh 2745 ($a/b = 2, e_{sx} = 0.20, \xi_s = 0.05, T_b = 2.5 \text{s}, e_{by} = 0, \Omega_b = 1$, and $e_{bx} = 0.15$).
After some algebraic manipulation, the vibration modes of the structure may be written as

\[
\Psi = \begin{bmatrix}
\tilde{\epsilon}_{bx} & \tilde{\epsilon}_{by} & 0 \\
\tilde{\epsilon}_{by} & -\tilde{\epsilon}_{bx} & 0 \\
0 & \omega_{01}^2 & \omega_{01}^2 - \omega_{02}^2
\end{bmatrix}.
\] (19)

Notice that the first column of \( \Psi \) corresponds to the normalized frequency \( \bar{\omega}_1 = 1 \), i.e., the uncoupled vibration frequency of the system \( \omega_1 = \omega_{by} \), and has the direction defined by the line connecting the CM and the CS of the building plan; the slope of this line is \( \bar{e}_{by}/\bar{e}_{bx} \) [12]. The first modal vector \( \Psi_1 \) is such that it coincides with the principal axis of stiffness, that is, a direction in which the restoring force of the system is aligned with the imposed floor displacement, and, hence, it only implies translation of the building plan. Because the normalized mass matrix in Eq. (17) is the \( 3 \times 3 \) identity matrix, the three modes of Eq. (19) are orthogonal as can be verified by simple observation of the inner product \( \Psi_i^T \Psi_j = 0 \) for \( i \neq j \). Furthermore, the 2D monosymmetric case is derived as a special case of Eqs. (18) and (19) by substitution of \( \bar{e}_{by} = 0 \), i.e., by redefining \( \delta = 1 + \Omega_b^2 + \bar{e}_{bx}^2 \) and contracting \( \Psi \) as

\[
\Psi_2 = \begin{bmatrix}
-\bar{\epsilon}_{bx} & -\bar{\epsilon}_{bx} \\
\bar{\epsilon}_{bx}/(1 - \bar{\omega}_2^2) & \bar{\epsilon}_{bx}/(1 - \bar{\omega}_3^2)
\end{bmatrix}
\] (20)

where lateral–torsional coupling is in the \( y-\theta \) direction. By using Eqs. (18)–(20) it is possible to estimate the modal displacements of the isolated base and superstructure through Eq. (15).

Although a well known aspect, it is important to revisit the concept of predominantly torsional and lateral frequencies of an asymmetric building. If a normalized system frequency, say in the \( y \)-direction, is larger than 1 while \( \Omega_b < 1 \), or vice versa, smaller than 1 while \( \Omega_b \geq 1 \), such a frequency is denoted as predominantly translational in the \( y \)-direction. The opposite definition is used for normalized rotational frequencies. The normalized values of the two vibration frequencies of a coupled monosymmetric single-story system, and the corresponding participation factors \( L_k^{(b)} \), are presented in Fig. 7 as a function of the normalized base eccentricity \( \bar{e}_{bx} \) and three values of \( \Omega_b = 0.8, 1.0, \) and 1.2. In this figure, the predominantly rotational frequencies are identified by solid lines and the translational ones by dashed lines. As it should be expected for \( \bar{e}_{bx} = 0 \), the normalized translational frequencies converge to 1, while the torsional frequencies converge to \( \Omega_b \). It is apparent that one of the frequencies increases with eccentricity while the other decreases in an almost linear fashion. Because the vibration frequencies move up and down the design spectrum from \( \omega_{yo} \), the shape of this spectrum in the neighborhood of \( \omega_{yo} \) has implications for the dynamic response of an asymmetric structure. This shift in the spectral ordinates needs to be considered in combination with the fact that the participation factors tend to be larger in general for the predominantly translational vibrations (Fig. 7)—with the exception of torsionally flexible structures with \( \Omega_b = 0.8 \) and \( \bar{e}_{bx} \geq 0.6 \).

Before computing the modal response represented by Eq. (15) for the 3D case, the modal participation factors are required. Using Eq. (19) with a bidirectional input motion in conjunction with the definition \( L_k^{(b)} = -\Psi^T [1 \ 0 \ 0 ; \ 0 \ 1 \ 0]^T \), the resulting modal participation factors are

\[
\Gamma = [\Gamma_{k,m}] = \begin{bmatrix}
\bar{e}_{bx}/\bar{\epsilon}_{bx}^2 & -\bar{e}_{by}/\bar{\epsilon}_{by}^2 & 0 \\
\bar{e}_{by}/\bar{\epsilon}_{by}^2 & \bar{e}_{bx}/\bar{\epsilon}_{bx}^2 & 0 \\
0 & \omega_{01}^2 & \omega_{01}^2 - \omega_{02}^2
\end{bmatrix}
\] (21)

where each column of \( \Gamma \) is associated with a different input. Therefore, the expected value of the maximum modal response of the 3D base motion for a bidirectional input is, using the summation convention in \( m \), \( \Gamma_{(k),m} A_{(k),m} \), where \( A_{(k),m} \) is the pseudo-acceleration value for the \( k \)th modal pair \( (\omega_k^{(b)}, \xi_k^{(b)}) \) and the \( m \)th input. For a single input component in the \( x \)-direction, the first column of \( \Gamma \) should be selected, while the second one corresponds to a ground motion in the \( y \)-direction. For instance, in the simpler 2D monosymmetric case, say for a ground motion in the \( y \)-direction, the modal participation factor...
would be

$$\mathbf{\Gamma} = \begin{bmatrix} \bar{\varepsilon}_{bx}/[\bar{\varepsilon}_{bx}^2 + (1 - \bar{\omega}_2^2)^2] & \bar{\varepsilon}_{by}/[\bar{\varepsilon}_{bx}^2 + (1 - \bar{\omega}_2^2)^2] \end{bmatrix}^T. \quad (22)$$

By substituting Eq. (21) in Eqs. (15) and (16), and assuming a ground motion in the y-direction, the maximum floor displacements of an isolated 3D superstructure can be estimated by the following expression

$$\mathbf{u}_o = \mathbf{f}(\varepsilon)^T \mathbf{m}(\varepsilon)^T \mathbf{r}(\varepsilon)$$

where the $\oplus$ sign is used to denote that each modal response cannot be added directly, but using the CQC presented in Eq. (16) and after pre-multiplying each term by $\mathbf{f}(\varepsilon)^T \mathbf{m}(\varepsilon)^T \mathbf{r}(\varepsilon)$. For a monosymmetric structure and input only in the y-direction, i.e., $\bar{\varepsilon}_{by} = 0$ and $\bar{\varepsilon}_{bx} \neq 0$, Eq. (23) can be simplified further to

$$\mathbf{u}_o = \mathbf{f}(\varepsilon)^T \mathbf{m}(\varepsilon)^T \mathbf{r}(\varepsilon)$$

For typical buildings, the term $\mathbf{m}(\varepsilon)^T \mathbf{r}(\varepsilon) = [\mathbf{m}_1 \mathbf{m}_2 \cdots \mathbf{m}_N]^T$ is a matrix with the floor masses piled up one on top of the other. Thus, the product of $\mathbf{m}(\varepsilon)^T \mathbf{r}(\varepsilon)$ and the term in brackets in Eq. (23) or (24) corresponds to the inertial forces as a result of the base motion, which pre-multiplied by the flexibility matrix $\mathbf{f}(\varepsilon)$ leads to the floor displacements. It is more important to notice that the coupling effect of the isolated base enters into the building response only through a torsional input characterized by the expressions in brackets in Eqs. (23) and (24).

6. Torsional amplification of base displacements

In this section torsional amplifications for the stiff and flexible edge displacements of the isolated base are analyzed. These results are presented first for monosymmetric structures, a wide range of base eccentricities $\bar{\varepsilon}_{bx}$, and uncoupled torsional-to-lateral frequency ratio $\Omega_b = 1.0$. Because the effect of the parameters of the superstructure on the base displacement amplifications is minor as long as the period ratio $T_s/T_b$ is small [6] (Fig. 8), the analysis considers an equivalent single-story superstructure with uncoupled fundamental frequency $T_s = 0.5 s$, static eccentricity $\bar{\varepsilon}_{sx} = 0$, and uncoupled torsional-to-lateral frequency ratios $\Omega_s = 0.8–1.2$. Indeed, the small width of the shaded bands in Fig. 8 implies that the eccentricity of the superstructure has a small effect on the torsional amplification of the base for a wide range of base eccentricities and $\Omega_s$.

The expected maximum response of the base can be computed from the CQC rule and the equations derived in the previous section. For instance, if the right and left edges are at distances $d_r$ and $d_l$ from the CM, the corresponding maximum edge displacements $q_r$ and $q_l$ for the monosymmetric case with $\bar{\varepsilon}_{by} = 0$ are

$$\begin{bmatrix} q_r \\ q_l \end{bmatrix} = \begin{bmatrix} \bar{\varepsilon}_{bx} + \frac{|d_r|}{\rho} (1 - \bar{\omega}_2^2) \\ \bar{\varepsilon}_{bx} - \frac{|d_l|}{\rho} (1 - \bar{\omega}_2^2) \end{bmatrix} \begin{bmatrix} \Omega_2 D_2 \\ \Omega_3 D_3 \end{bmatrix}$$

where the $d_i/r$ term corrects the normalization introduced in Eq. (17); and $D_2$ and $D_3$ are the spectral displacements corresponding to the second and third vibration modes. Results of Eq. (25) are presented in Figs. 9 and 10 for individual ground motions, averages for the suite of records (Table 1), and the UBC design spectrum. Solid lines correspond to the torsional amplification for the flexible edge and dashed lines to the amplification of the stiff edge.

Parts (a) and (b) of Fig. 9 show the amplifications of the isolated base for the El Centro (1940) and Melipilla (Chile, 1985) ground motions. It is apparent that the amplification for the flexible edge tends to increase with the eccentricity and is considerably larger for the Melipilla record, with values that range from 1 to 3 as the base eccentricity increases. Nevertheless, mean amplifications at the flexible edge shown in part (c) rarely exceed 1.5 for a wide range of eccentricities and records. Indeed, this result is also validated by the amplifications derived from probabilistic analysis in part (d). Notice also that the flexible edge displacement amplification
is moderately sensitive with $\Omega_b$. However, the effect of $\Omega_b$ is more notorious for the stiff edge of the plan; the stiffer the plan in torsion, the smaller the stiff edge displacement amplification. For instance, with $\Omega_b = 0.8$ and small eccentricities, the stiff edge displacements are amplified to a maximum of about 1.2. Otherwise, the stiff edge displacements are usually deamplified and decrease as the eccentricity of the isolated base increases. Fig. 10 shows a comparison of the use of dynamic and static analysis according to the UBC spectrum and code. Static amplifications are derived from the well known UBC [16] code formula $\bar{q}_e = 1 + e_{bx} x / \rho^2$ and are compared against the dynamic amplifications computed from Eq. (25) for the UBC displacement spectrum. In general, edge displacement amplifications show similar trends to the ones presented in Fig. 9. However, flexible edge amplifications using the UBC spectrum are larger than the average results obtained for the suite of records. On the other hand, the code static formula leads to a conservative bound for the amplification expected; however, it does not recognize the effect of the torsional-to-lateral frequency ratio, since it assumes $\Omega_b = 1$. A modified expression that incorporates the effect of the torsional-to-lateral frequency ratio, $\kappa$, is as follows: $\bar{q}_e = 1 + e_{bx} x / k^2$, with $k^2 = k_{g}/k_{v}$ the radius of gyration of stiffness. Thus, the code formula assumes the approximation $\kappa = \rho$, i.e., the stiffness radius of gyration equal to the mass radius of gyration.

Shown in Fig. 11 is the effect caused by the shape of the displacement spectrum on the maximum torsional amplification for all possible eccentricities, $-a/2 \leq e_{bx} \leq a/2$. It is apparent that the stiff edge displacement amplification is little sensitive to the shape of the spectrum, and amplifications reach a maximum of about 1.4 for $\Omega_b = 0.8$. However, if the structure falls in the constant pseudo-velocity region, or constant displacement region, or in both regions, the torsional amplifications may change drastically. Indeed, for the constant pseudo-velocity region, amplifications may be as large as 3.5. Such amplifications decrease to a maximum of 1.5 if the vibration frequencies fall into the constant displacement...
Table 2
Fixed-base and isolated periods and modes of the six-story building example

<table>
<thead>
<tr>
<th>Story</th>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>T&lt;sub&gt;b&lt;/sub&gt; (s)</td>
<td>T&lt;sub&gt;s&lt;/sub&gt; (s)</td>
<td>θ</td>
<td>θ</td>
<td>θ</td>
</tr>
<tr>
<td>6</td>
<td>y</td>
<td>0.5827</td>
<td>−0.5945</td>
<td>−0.5459</td>
<td>−0.5252</td>
<td>0.4666</td>
</tr>
<tr>
<td></td>
<td>θ</td>
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<td>−0.0018</td>
<td>−0.0018</td>
<td>0.0006</td>
<td>0.0015</td>
</tr>
<tr>
<td>5</td>
<td>y</td>
<td>0.5232</td>
<td>−0.5309</td>
<td>−0.1890</td>
<td>−0.1871</td>
<td>−0.2526</td>
</tr>
<tr>
<td></td>
<td>θ</td>
<td>0.0018</td>
<td>0.0006</td>
<td>−0.0006</td>
<td>0.0002</td>
<td>−0.0008</td>
</tr>
<tr>
<td>4</td>
<td>y</td>
<td>0.4523</td>
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<td>0.2389</td>
<td>0.2158</td>
<td>−0.5502</td>
</tr>
<tr>
<td></td>
<td>θ</td>
<td>0.0015</td>
<td>0.0005</td>
<td>0.0008</td>
<td>−0.0003</td>
<td>−0.0017</td>
</tr>
<tr>
<td>3</td>
<td>y</td>
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<td>−0.3360</td>
<td>0.5210</td>
<td>0.5065</td>
<td>−0.0932</td>
</tr>
<tr>
<td></td>
<td>θ</td>
<td>0.0012</td>
<td>0.0004</td>
<td>0.0017</td>
<td>−0.0006</td>
<td>−0.0002</td>
</tr>
<tr>
<td>2</td>
<td>y</td>
<td>0.2165</td>
<td>−0.2167</td>
<td>0.5237</td>
<td>0.5430</td>
<td>0.4851</td>
</tr>
<tr>
<td></td>
<td>θ</td>
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<td>0.0002</td>
<td>0.0017</td>
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<td>0.0016</td>
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<tr>
<td>1</td>
<td>y</td>
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<td>−0.0925</td>
<td>0.2576</td>
<td>0.3019</td>
<td>0.4144</td>
</tr>
<tr>
<td></td>
<td>θ</td>
<td>0.0003</td>
<td>0.0001</td>
<td>0.0008</td>
<td>−0.0003</td>
<td>0.0013</td>
</tr>
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<td>Base</td>
<td>T&lt;sub&gt;b&lt;/sub&gt; (s)</td>
<td>2.96</td>
<td>2.36</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>y</td>
<td>−0.5696</td>
<td>−0.8219</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>θ</td>
<td>−0.8219</td>
<td>0.5696</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Fig. 11. Effect of the spectral shape on the torsional amplifications 〈q〉 at both building edges for a structure defined as 〈e<sub>xx</sub>〉 = 0, 〈ζ<sub>ε</sub>〉 = 0.05; T<sub>s</sub> = 0.5 s, a/b = 2, 〈ζ<sub>ε</sub>〉 = 0.15, and subjected to the NCh 2745 design spectrum.

region, or in the combined case. Consequently, the shape of the spectrum determines the magnitude of the torsional amplifications observed. This may explain in part why some earlier researchers found contradictory results regarding this amplification.

For the sake of completeness, the 3D mean torsional amplifications at each of the four corners of a rectangular plan building and computed with the EXM are summarized in Fig. 12. The system parameters are T<sub>b</sub> = 2.5 s, T<sub>s</sub> = 0.5 s, 〈ζ<sub>ε</sub>〉 = 0.15, 〈ζ<sub>ε</sub>〉 = 0.05, and Ω<sub>ε</sub> = 1.0. Building corners are numbered counterclockwise from 1 to 4 starting from the bottom left. Each point in this figure represents average values for eight records (Table 1) of the maximum resultant displacement at the corner normalized by the maximum resultant displacement at the CM. The structures considered include eccentricity in both horizontal directions of analysis, 〈e<sub>by</sub>〉 = 0, 0.2, and 0.4, and two values of uncoupled frequency ratio Ω<sub>b</sub> = 0.8, and 1.2. In general, the mean peak amplifications for corner displacements are usually smaller than 1.6, which is similar to the factor obtained for monosymmetric structures. Although not always, mean displacement amplifications tend to be larger for the smaller uncoupled frequency ratio, Ω<sub>b</sub> = 0.8. Positive and larger normalized eccentricities, 〈e<sub>bx</sub>〉 and 〈e<sub>by</sub>〉, tend to produce an increase in displacement at corners 1 and 4, while negative and larger eccentricities do the same at corners 2 and 3. For 〈e<sub>by</sub>〉 positive and Ω<sub>b</sub> = 1.2, mean displacement amplifications at corners 1 and 2 increase with larger 〈e<sub>by</sub>〉, while the opposite occurs at corners 3 and 4. Although the effect of bidirectional eccentricity and motion in the mean amplifications is not large, their values show dependency with 〈e<sub>by</sub>〉 and Ω<sub>b</sub>. Consequently, a general and accurate expression to avoid the use of Eq. (25) seems infeasible. A comparison of these results with those of parts (c) and (d) of Fig. 9 shows that the effect of 〈e<sub>by</sub>〉 on the torsional amplification of the same edge displacements is small. Therefore, this suggests that both lateral dynamic analyses in the two principal directions could be performed independently.

7. Design example

To demonstrate the application of the above expressions in multistory building design, let us consider the seismic response of an asymmetric six-story R/C structure that has been base isolated (Fig. 13). The building is a three-dimensional moment resistant frame of rectangular plan, 1500 × 1000 mm, plan aspect ratio a/b = 3/2, and seismic weight w = 1 ton/m<sup>2</sup>. Beams and columns in the superstructure are regularly distributed with respect to the CM; the stiffness eccentricity present in the superstructure is a result of the asymmetric steel diagonal bracing shown in the figure. A summary of the structural properties for this example is presented in Fig. 13. The superstructure has a fundamental vibration period T<sub>s</sub> = 0.68 s, uncoupled torsional-to-lateral frequency ratio Ω<sub>ε</sub> = 0.8, normalized static eccentricity 〈e<sub>xx</sub>〉 = 0.25, and constant modal damping ratio 〈ζ<sub>ε</sub>〉 = 0.05. The first five natural frequencies of the fixed-base superstructure are presented in Table 2. On the other hand, the isolation system is characterized by the uncoupled translational period T<sub>b</sub> = 2.5 s, uncoupled torsional-to-lateral frequency ratio Ω<sub>b</sub> = 0.9, normalized static eccentricity 〈e<sub>bx</sub>〉 = −0.2, and constant modal
Fig. 12. Average corner torsional amplification for bidirectional input as a function of $\vec{e}_{bx}$ and eccentricity $\vec{e}_{by} = 0, 0.2, \text{ and } 0.4$, with the superstructure defined as $a/b = 2$, $e_{xx} = e_{yy} = 0$, $s = 0.05$, $T_s = 0.5$ s and $s = 1$.

Fig. 13. Building plan and elevation of the six-story building example $a/b = 3/2$, $e_{xx} = 0.25$, $\zeta_s = 0.05$, $T_s = 0.68$ s, $\omega_s = 0.8$, $e_{bx} = -0.20$, $\zeta_b = 0.15$, $T_b = 2.5$ s, and $\Omega_b = 0.9$. 
damping ratio $\zeta_b = 0.15$. The natural vibration frequencies and modes for the isolated base are included in Table 2.

The procedure for estimating the design displacements at the stiff and flexible edges has the following steps: (1) compute the vibration frequencies and modes of the isolated base with rigid superstructure by using Eqs. (18) and (19); (2) compute the modal participation factors by Eq. (21); (3) compute the reduced spectral ordinates $D_k^{(b)}$ and $A_k^{(b)} = \omega_k^{(b)^2} D_k^{(b)}$ from the design spectrum with $\zeta_b = 0.15$ damping ratio; (4) evaluate the edge displacements $q_r$ and $q_l$ from Eq. (25); (5) evaluate the modal displacement response of the superstructure using Eq. (13); and (6) compute the edge displacements and torsional amplifications in the superstructure. In this example, the NCh2745 design spectrum [11] for stiff soil and a compatible record with this spectrum are used in the analysis. Notice that, as opposed to the UBC [16] spectrum with constant pseudo velocity, the NCh2745 spectrum has constant displacement for isolated periods $T_b \geq 2$ s.

For the sake of brevity, only steps (1) through (4) are presented in detail next. After substituting $\delta = 1 + \frac{\Omega_b^2}{\Delta_b^2} = 1 + 0.9^2 + 0.20^2 = 1.85$, step (1) leads to the following system frequencies and modes

$$\omega = \begin{bmatrix} 2.51 & 2.68 & 2.12 \end{bmatrix} \text{rad/s and}$$

$$\Psi = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & 0.2 & 0.2 \\ 0 & -0.139 & 0.289 \end{bmatrix}.$$  \hspace{1cm} (26)

Hence, the normalized frequencies are $\tilde{\omega}_k = [1 \ 1.07 \ 0.84]^T$. Besides, by using Eq. (21) in step (2), the modal participation factors are $\Gamma_k = [-0 \ 3.38 \ 1.62]^T$. Also, since the design spectrum for $\zeta_b = 0.05$ is constant for $T_b \geq 2$ s and equal to $D_k^{(b)} = 30$ cm, step (3) leads to the $\zeta_b = 0.05$ spectral acceleration ordinates $A_k^{(b)} = \omega_k^{(b)^2} D_k^{(b)} = [189.5 \ 216.0 \ 134.9] \text{cm/s}^2$. These values, and the spectral displacements $D_k$, need to be reduced for the isolation damping ratio $\zeta_b = 0.15$. Using the expression for the reduction factor $B_k$ contained in NCh2745, the corresponding reductions are constant and equal to $B_k = 1.67$. Hence, the corresponding corrected spectral displacements and accelerations are $D_k = 30/1.67 = 17.95$ cm, and $A_k = \sqrt{1 + 4\tilde{\omega}_k^2 D_k^{(b)}}/B_k = [118.4 \ 134.9 \ 84.3] \text{cm/s}^2$. Finally, the right and left edge displacements $q_r$ and $q_l$ at the base, which correspond to the flexible and stiff edge responses, respectively, are evaluated from Eq. (25), i.e.,

$$\begin{bmatrix} q_r \\ q_l \end{bmatrix} = \begin{bmatrix} -\tilde{e}_{bh} + \frac{|d|}{\rho} \left(1 - \tilde{\omega}_k^2\right) \Gamma_2 D_2 \\ -\tilde{e}_{bh} - \frac{|d|}{\rho} \left(1 - \tilde{\omega}_k^2\right) \Gamma_2 D_2 \end{bmatrix} \oplus \begin{bmatrix} -\tilde{e}_{bh} + \frac{|d|}{\rho} \left(1 - \tilde{\omega}_k^2\right) \Gamma_3 D_3 \\ -\tilde{e}_{bh} - \frac{|d|}{\rho} \left(1 - \tilde{\omega}_k^2\right) \Gamma_3 D_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3.38 [0.2 + 1.44(-0.139)][17.9] \oplus 1.62 [0.2 + 1.44(0.289)][17.9] \\ 3.38 [0.2 - 1.44(-0.139)][17.9] \oplus 1.62 [0.2 - 1.44(0.289)][17.9] \end{bmatrix}$$

$$= \begin{bmatrix} 17.94 \\ 20.97 \end{bmatrix} \text{cm.} \hspace{1cm} (27)$$

Consequently, the amplification factors obtained for the stiff and flexible edges with respect to the symmetric case are $20.97/(30/1.67) = 1.17$ and $17.94/(30/1.67) = 1$, respectively. As should be the case for torsionally flexible structures [14], these results show that the larger displacement amplification occurs along the stiff edge. It is interesting to note that the UBC code formula [16] would predict an amplification of 1.33 for the flexible edge and a reduction factor of 0.75 for the stiff edge, which are not only numerically incorrect but inconsistent conceptually. This simple example shows that the static code formula does not capture the true dynamics of the system.

Fig. 14 shows the response history of the building example subjected to the NCh2745 compatible earthquake record. Displacements at both building edges are presented. Superimposed on these traces are the peak response values determined manually as presented in the previous paragraph. Notice that the predicted peak responses in Eq. (27) are very good approximations of the peak seismic response at both building edges and should be used instead of the UBC code formula, which, based on a crude static approximation, overestimates the edge response of the flexible edge and significantly underestimates the response of the stiff edge. Although shown for a single example, this observation carries over to many other cases [14].

Only for the sake of completeness, Table 3 shows a comparison of the superstructure edge displacements obtained with the proposed procedure and exact dynamic analysis. Exact values are shown in parentheses in the last column of the table. The overall accuracy of the proposed procedure to estimate the edge displacements is very good, leading to a maximum error of 7%. This good accuracy of the procedure can be advantageously used in the design of the building since it only requires the isolated based modes and frequencies together with equivalent lateral analysis of the superstructure.

8. Conclusions

This investigation uses the dynamic interaction between the base and superstructure to study the seismic behavior of plan asymmetric buildings. As conventionally done, such behavior was studied in terms of the observed amplification of displacements at the stiff and flexible edges of the building plan. It was shown that the amplifications can be conservatively approximated by a model that includes a mass correction to
account for the dynamic interaction between the base and superstructure. The MCM is capable of reproducing almost exactly the vibration frequencies of the building and leads to conservative estimates of the building response.

Additionally, a very simple quasi-static model (QSM) was derived to approximate the response of the superstructure. The physical justification for this model is based on the usually large ratio between the fixed-base and isolated frequencies for most practical isolated buildings. Hence, for a wide range of structural configurations, the superstructure perceives the motions of the base essentially as quasi-static. This enables the design engineer to estimate the displacements of the superstructure by considering three vectors of equivalent inertial forces, each corresponding to one of the modes of the isolated base. Results for the base have shown that the displacements estimated according to the approximate procedure presented herein are usually within 10% error. As opposed to this, the displacement amplifications predicted by the UBC-code formula grossly overestimate the true dynamic amplifications on the isolated base, specially for small values of the lateral-to-torsional frequency ratio of the superstructure.

Numerical results have shown that the UBC code formula, which is based on a static approximation, does not lead to accurate and conservative edge displacement approximations. Indeed, for torsionally flexible isolation systems $\Omega_b < 1$ and base eccentricities $\tilde{e}_b \leq 0.7$, the maximum amplification of the response occurs at the stiff edge and is usually smaller than 1.5. Otherwise, for torsionally stiff isolation systems $\Omega_b \geq 1.0$, the maximum amplifications occur at the flexible edge. The closed form expressions developed for modal analysis of the base are shown to provide accurate results and are proposed to replace the UBC static code formula. These expressions were applied to a six-story building example and showed the simplicity in estimating the torsional amplification for the edge displacements of the isolated base, a crucial aspect in designing an isolation system and the corresponding gap. The usefulness of this formulation resides not only in its numerical simplicity, but also on the conceptual understanding of the three-dimensional behavior of plan asymmetric isolated structures. An important intuitive result derives from the observation that the base–superstructure coupling has only a minor effect on the base response and, hence, it is possible to adjust the isolated base parameters to control the torsional response of the superstructure without causing an imbalance of the edge deformations of the base.

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References


